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RECEPTANCE MATRICES OF VISCOUSLY DAMPED SYSTEMS SUBJECT TO SEVERAL CONSTRAINT EQUATIONS

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1. INTRODUCTION

As is known, the receptance matrix (also called the frequency response matrix) is an important matrix which interrelates the input and output of a damped linear discrete mechanical system which is subject to harmonical forcing as input. There are many publications in the literature on this subject. Some of the recent publications are references [1–3]. Yang presented in reference [1] an exact method for evaluating the receptances of non-proportionally damped dynamic systems. Based on a decomposition of the damping matrix, an iteration procedure is developed which does not require matrix inversion. In reference [2], Lin *et al.* developed a new and effective method to derive structural design sensitivities, which include both frequency response function sensitivities and eigenvalue and eigenvector sensitivities from limited vibration test data. The study of Mottershead [3] was concerned with the zeros of structural frequency response functions and their sensitivities.

The present study is concerned with a viscously damped linear discrete mechanical system the co-ordinates of which are assumed to be subject to several linear constraint equations.

In a series of papers, the present author has investigated the characteristic equation of such constrained systems [4–6]. The main aim here is the establishment of the frequency response matrix of the constrained system described above in terms of the frequency response matrix of the unconstrained system and the coefficient vectors of the constraint equations.

2. THEORY

The motion of a viscously damped linear discrete mechanical system with n degrees of freedom which is harmonically excited, is governed in the physical space by the matrix differential equation of order two

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \bar{\mathbf{F}}\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

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where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the $(n \times n)$ mass, damping and stiffness matrices respectively. $\mathbf{q}$ is the $(n \times 1)$ vector of generalized co-ordinates. $\bar{\mathbf{F}}$ is the forcing vector and ω denotes the forcing frequency.

Substitution of

$$\mathbf{q}(t) = \bar{\mathbf{q}}\mathrm{e}^{\mathrm{i}\omega t} \tag{2}$$

into equation (1) yields the relation

$$\bar{\mathbf{q}} = \mathbf{H}(\omega)\bar{\mathbf{F}} \tag{3}$$

between the constant part of the input and response vectors. The complex matrix

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{D} + \mathbf{K})^{-1} \tag{4}$$

is referred to as the (complex) frequency response matrix or the receptance matrix. It is also referred to as the admittance matrix or dynamic influence coefficient matrix [7].

Now assume that the co-ordinates of the mechanical system are subject to linear constraint equations of the form

$$\mathbf{a}_p^{\mathrm{T}}\mathbf{q} = 0, \quad p = 1, \ldots, l, \tag{5}$$

where the pth vector of constraint coefficients is defined as $\mathbf{a}_p^{\mathrm{T}} = [a_{1p}, \ldots, a_{np}]$.

The main concern of the present study is to establish the receptance matrix of the constrained system described above.

By means of the Lagrange's equations formalism in connection with Lagrange's multipliers, equations (1) and (5) can be combined as [8]

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \sum_{j=1}^{l} \mu_j\mathbf{a}_j + \mathbf{F}_j\mathrm{e}^{\mathrm{i}\omega t}, \tag{6}$$

where μ_j denotes the corresponding Lagrange multiplier. If harmonical solutions of form (2) and

$$\mu_j = \bar{\mu}_j\mathrm{e}^{\mathrm{i}\omega t} \tag{7}$$

are substituted into equation (6)

$$\bar{\mathbf{q}} = \sum_{j=1}^{l} \bar{\mu}_j\mathbf{H}\mathbf{a}_j + \mathbf{H}\bar{\mathbf{F}} \tag{8}$$

is obtained. After substitution of $\bar{\mathbf{q}}$ into the constrained equations (5), the following equations are obtained for the determination of the unknown amplitudes $\bar{\mu}_j$

$$[\mathbf{a}_p^{\mathrm{T}}\mathbf{H}\mathbf{a}_1]\bar{\mu}_1 + \cdots + [\mathbf{a}_p^{\mathrm{T}}\mathbf{H}\mathbf{a}_l]\bar{\mu}_l = -\,\mathbf{a}_p^{\mathrm{T}}\mathbf{H}\bar{\mathbf{F}}, \quad p = 1, \ldots, l. \tag{9}$$

From this set of l inhomogeneous equations for $\bar{\mu}_p$,

$$\bar{\boldsymbol{\mu}} = -\,\mathbf{A}_{ll}^{-1}\mathbf{A}_{ln}\mathbf{H}\bar{\mathbf{F}} \tag{10}$$

is obtained where the following definitions are introduced:

$$\bar{\boldsymbol{\mu}} = [\bar{\mu}_1, \ldots, \bar{\mu}_l]^{\mathrm{T}}, \quad \mathbf{A}_{ln}^{\mathrm{T}} = [\mathbf{a}_1, \ldots, \mathbf{a}_l], \quad \bar{\mathbf{F}} = [\bar{F}_1, \ldots, \bar{F}_n]^{\mathrm{T}} \tag{11}$$

and the (p, q)th element of the $(l \times l)$ matrix $\mathbf{A}_{ll}$ is defined as $\mathbf{a}_p^{\mathrm{T}}\mathbf{H}\mathbf{a}_q$.

Using the above definitions, expression (8) can be reformulated as,

$$\bar{\mathbf{q}} = \mathbf{H}\mathbf{A}_{ln}^{\mathrm{T}}\bar{\boldsymbol{\mu}} + \mathbf{H}\bar{\mathbf{F}}. \tag{12}$$

The substitution of $\bar{\boldsymbol{\mu}}$ from equation (10) into equation (12) yields

$$\bar{\mathbf{q}} = \mathbf{H}[\mathbf{I} - \mathbf{A}_{ln}^{\mathrm{T}}\mathbf{A}_{ll}^{-1}\mathbf{A}_{ln}\mathbf{H}]\bar{\mathbf{F}}, \tag{13}$$

which in turn gives the receptance matrix of the constrained system in terms of the receptance matrix of the unconstrained system and the vectors of constraint coefficients, in the form

$$\mathbf{H}_{cons} = \mathbf{H}[\mathbf{I} - \mathbf{A}_{ln}^{\mathrm{T}}\mathbf{A}_{ll}^{-1}\mathbf{A}_{ln}\mathbf{H}]. \tag{14}$$

$\mathbf{I}$ is the $(n \times n)$ unit matrix. In the special case of only one constraint equation, i.e., $l = 1$ the receptance matrix simplifies to

$$\mathbf{H}_{cons} = \mathbf{H}\left[\mathbf{I} - \frac{\mathbf{a}_1\mathbf{a}_1^{\mathrm{T}}\mathbf{H}}{\mathbf{a}_1^{\mathrm{T}}\mathbf{H}\mathbf{a}_1}\right]. \tag{15}$$

3. NUMERICAL EVALUATIONS

This section is devoted to the testing of the reliability of the expressions obtained. The simple system in Figure 1 is taken as an illustrative example. It consists of a vibrational system with three degrees of freedom, viscously damped at the first mass. Each mass m_i is acted upon by a harmonically varying force F_i, with the forcing frequency ω. The system matrices are as follows:

$$\mathbf{M} = \mathbf{diag}(m_1, m_2, m_3), \quad \mathbf{D} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -\,k_2 & 0 \\ -\,k_2 & k_2 + k_3 & -\,k_3 \\ 0 & -\,k_3 & k_3 \end{bmatrix}. \tag{16}$$

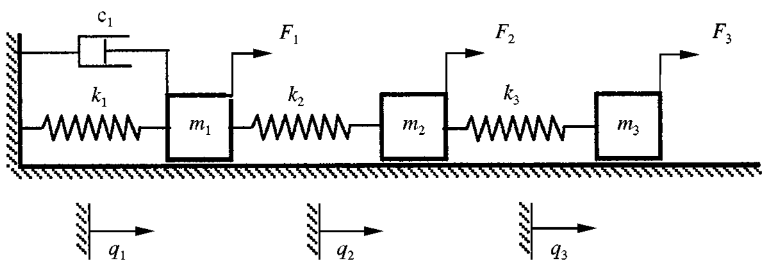

Figure 1. The unconstrained three-degrees-of-freedom system used as the sample.

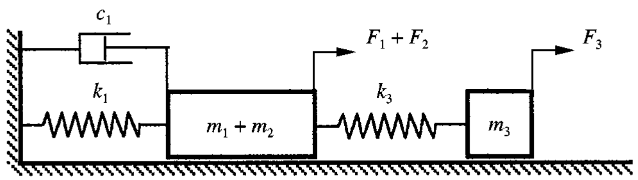

Figure 2. The mechanical system obtained from the system in Figure 1 by imposing the constraint $q_1 = q_2$.

Assume that the following numerical values are chosen for the physical parameters of the system: $k_1 = 10$ N/m, $k_2 = 10$ N/m, $k_3 = 20$ N/m, $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 4$ kg, $c_1 = 10$ N/m/s, $\omega = 2$ rad/sec.

The receptance matrix given in equation (4) is obtained as

$$\mathbf{H} = \begin{bmatrix} 0{\cdot}0247 - 0{\cdot}0286\mathrm{i} & -0{\cdot}0032 + 0{\cdot}0037\mathrm{i} & -0{\cdot}0159 + 0{\cdot}0183\mathrm{i} \\ -0{\cdot}0032 + 0{\cdot}0037\mathrm{i} & -0{\cdot}0124 - 0{\cdot}0005\mathrm{i} & -0{\cdot}0621 - 0{\cdot}0024\mathrm{i} \\ -0{\cdot}0159 + 0{\cdot}0183\mathrm{i} & -0{\cdot}0621 - 0{\cdot}0024\mathrm{i} & -0{\cdot}0603 - 0{\cdot}0118\mathrm{i} \end{bmatrix}. \tag{17}$$

Now, assume that the system in Figure 1 is, due to some reason, subject to a constraint like $q_1 = q_2$. The mechanical system so constrained is shown in Figure 2. It is desired to obtain the receptance matrix of this system.

It is easy to see that the constraint coefficient vector is

$$\mathbf{a}_1 = [1 \ \ -1 \ \ 0]^{\mathrm{T}}. \tag{18}$$

Denote the properties of the constrained system by the subscript "cons". The corresponding matrices and vectors are

$$\mathbf{M}_{cons} = \begin{bmatrix} (m_1 + m_2) & 0 \\ 0 & m_3 \end{bmatrix}, \qquad \mathbf{D}_{cons} = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{K}_{cons} = \begin{bmatrix} (k_1 + k_3) & -k_3 \\ -k_3 & k_3 \end{bmatrix}, \tag{19}$$

$$\mathbf{q}_{cons} = \bar{\mathbf{q}}_{cons} e^{i\omega t} = \begin{bmatrix} \bar{q}_{1cons} \\ \bar{q}_{3cons} \end{bmatrix} e^{i\omega t}, \qquad \mathbf{F}_{cons} = \bar{\mathbf{F}}_{cons} e^{i\omega t} = \begin{bmatrix} (\bar{F}_1 + \bar{F}_2) \\ \bar{F}_3 \end{bmatrix} e^{i\omega t},$$

$$\bar{\mathbf{q}}_{cons} = \mathbf{H}_{cons} \bar{\mathbf{F}}_{cons},$$

where

$$\mathbf{H}_{cons} = (-\omega^2 \mathbf{M}_{cons} + i\omega \mathbf{D}_{cons} + \mathbf{K}_{cons})^{-1}. \tag{20}$$

If the numerical values chosen are substituted above, the receptance matrix of the constrained system is obtained as

$$\mathbf{H}_{cons} = -\begin{bmatrix} 0{\cdot}0115 + 0{\cdot}0028i & 0{\cdot}0576 + 0{\cdot}0140i \\ 0{\cdot}0576 + 0{\cdot}0140i & 0{\cdot}0378 + 0{\cdot}0702i \end{bmatrix}. \tag{21}$$

On the other hand, substituting equations (17) and (18) into formula (15) of the present study yields

$$\mathbf{H}_{cons} = -\begin{bmatrix} 0{\cdot}0115 + 0{\cdot}0028i & 0{\cdot}0115 + 0{\cdot}0028i & 0{\cdot}0576 + 0{\cdot}0140i \\ 0{\cdot}0115 + 0{\cdot}0028i & 0{\cdot}0115 + 0{\cdot}0028i & 0{\cdot}0576 + 0{\cdot}0140i \\ 0{\cdot}0576 + 0{\cdot}0140i & 0{\cdot}0576 + 0{\cdot}0140i & 0{\cdot}0378 + 0{\cdot}0702i \end{bmatrix}. \tag{22}$$

The fact that the dimensions of $\mathbf{H}_{cons}$ are different in equations (21) and (22) should cause no confusion. It is clearly seen that both forms of the receptance matrices lead to the same input–output relations:

$$\begin{aligned} \bar{q}_1 &= -(0{\cdot}0115 + 0{\cdot}0028i)(\bar{F}_1 + \bar{F}_2) - (0{\cdot}0576 + 0{\cdot}0140i)\bar{F}_3, \\ \bar{q}_2 &= \bar{q}_1, \\ \bar{q}_3 &= -(0{\cdot}0576 + 0{\cdot}0140i)(\bar{F}_1 + \bar{F}_2) - (0{\cdot}0378 + 0{\cdot}0702i)\bar{F}_3. \end{aligned} \tag{23}$$

4. CONCLUSIONS

This study is concerned with a viscously damped linear discrete mechanical system which is excited harmonically. The co-ordinates of the system are assumed to be subject to several linear constraint equations. The main concern is the establishment of the receptance matrix of the so constrained system in terms of the receptance matrix of the unconstrained system and the coefficient vectors of the constraint equations.

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